A REMARK ON THE

"BANG-BANG" PRINCIPLE FOR LINEAR CONTROL SYSTEMS

IN INFINITE DIMENSIONAL SPACE

by

H. O. Fattorini*

Center for Dynamical Systems
Brown University
Providence, Rhode Island

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INTRODUCTION

The "bang-bang" principle for linear control systems

$$u'(t) = A(t)u(t) + B(t)f(t) \qquad (1)$$

in finite-dimensional space E^n (J. P. LaSalle, [1]) can be stated as follows: if the system (1) can be steered from a point $u \in E^n$ to another point $v \in E^n$ in a given time $t_1 > 0$ by a control f taking values, say, in the unit cube K of E^m then the transfer of (1) from u to v can also be achieved in the same time by another control f_0 taking values in K_0, the set of extremal points of K. This result has been extended in various directions; let us only mention [9], where K is allowed to be any compact convex set in E^m. (See also [4], [7] for other types of generalizations). The "bang-bang" principle does not hold in infinite-dimensional spaces; in fact, it is easy to construct control systems, even with A and B time-independent where the final state v at a given time t_2 depends uniquely on the control f. (See [2], [3]). However, the principle subsists if we satisfy ourselves with approximating (and not actually attaining) the final state. Moreover, it turns out that we can also approximate the whole trajectory between u and v by means of K_0-valued controls (Theorem 2.2). This result is similar in form to the one in [5] for non-linear control systems in finite-dimensional space; its proof is an application of elementary facts of the theory of integration of vector-valued functions.

1. THE INITIAL-VALUE PROBLEM

We shall denote by E, F two (real or complex) Banach spaces, $L(F;E)$ the Banach space of all linear bounded operators from F to E endowed, as usual, with the uniform operator topology. For each t in $[t_o, t_1]$, $t_o < t_1$ $A(t)$ will be a (possibly unbounded) linear operator with domain $D(A(t))$. We shall assume that the Cauchy problem for

$$u'(t) = A(t)u(t) \tag{1.1}$$

is well set. This means there exists an evolution operator $U(t,s)$, i.e. a strongly continuous $L(E,E)$-valued function $U(t,s)$ defined in the triangle $t_o \leq s \leq t \leq t_1$ satisfying $U(t,t) = I$, $t_o \leq t \leq t_1$ and such that for each $t \in [t_o, t_1)$, $u \in E$ the function

$$u(s) = U(s,t)u$$

is a (classical or generalized) solution of (1.1) in $[t, t_1]$. For any strongly measurable function $g(\cdot)$ defined and summable in $[t_o, t_1]$ and any $u \in E$ we shall define the expression

$$u(t) = U(t,t_o)u + \int_{t_o}^{t} U(t,s)g(s)ds \tag{1.2}$$

to be a solution of the inhomogeneous equation

$$u'(t) = A(t)u(t) + g(t). \tag{1.3}$$

It is easy to see that the function $u(\cdot)$ defined by (1.2) exists and is continuous in $[t_o, t_1]$. Under additional conditions on $A(t), U(t,s), g(s)$

and u it is possible to show that (1.2) is a genuine solution of (1.3); we shall not dwell upon this point here.

Finally, we consider the linear control system

$$u'(t) = A(t)u(t) + B(t)f(t), \quad t_0 \leq t \leq t_1 \tag{1.4}$$

For each t, $t_0 \leq t \leq t_1$, $B(t)$ is a bounded operator from F to E. We assume that $B(\cdot)$ is strongly measurable, i.e. that for any $u \in F$ the E-valued function $B(\cdot)u$ is strongly measurable; moreover, we suppose there exists a scalar-valued function $\eta(\cdot)$, summable in $[t_0,t_1]$ such that

$$|B(t)| \leq \eta(t), \quad t_0 \leq t \leq t_1 \tag{1.5}$$

The class of controls $\mathcal{L}_K$ consists of all strongly measurable F-valued functions $f(\cdot)$ defined in $[t_0,t_1]$ with values in some fixed closed, bounded, convex set K. The <u>trajectories</u> of the system (1.4) are the solutions of (1.4) for some control $f \in \mathcal{L}_K$, i.e. functions of the form

$$u(t) = U(t,t_0)u + \int_{t_0}^{t} U(t,s)B(s)f(s)ds \tag{1.6}$$

with $f \in \mathcal{L}_K$. Since $B(\cdot)f(\cdot)$ is summable in E, each trajectory $u(\cdot)$ is continuous in $[t_0,t_1]$.

2. <u>THE BANG-BANG PRINCIPLE</u>

In all that follows, K_0 will be a subset of K satisfying

2.1 <u>ASSUMPTION</u>. Finite convex combinations of elements of K_0 (i.e. finite sums $\Sigma \lambda_k u_k$, $\lambda_k \geq 0$, $\Sigma \lambda_k = 1$, $u_k \in K_o$) are dense in K.

Let us call $\mathcal{L}_{K_o}$ the subset of $\mathcal{L}_K$ defined by the following two conditions:

(a) There exists disjoint intervals $I_1, \ldots, I_n, \bigcup I_k = [t_o, t_1]$ such that f is constant in each I_k.

(b) $f(t) \in K_o$ for all $t \in [t_o, t_1]$.

2.2 THEOREM Let $u(\cdot)$ be a trajectory of (1.4) corresponding to some $f \in \mathcal{L}_K$ and let $\epsilon > 0$. Then there exists a $f_o \in \mathcal{L}_{K_o}$ such that the trajectory $u_o(\cdot)$ corresponding to f_o satisfies

$$|u(t) - u_o(t)|_E \leq \epsilon, \quad t_o \leq t \leq t_1.$$

The proof of Theorem 2.2 is a consequence of the following auxiliary result:

2.3. LEMMA Let X be a Banach space, $N(\cdot)$ a $L(F; X)$-valued, strongly measurable function defined in $[t_o, t_1]$ such that $|N(t)| \leq \eta(t)$, $t_o \leq t \leq t_1$ for some summable function $\eta(\cdot)$. Let $\mathcal{K}$ ($\mathcal{K}_o$) be the set of all elements of X of the form

$$\int_{t_o}^{t_1} N(s)f(s)ds, \tag{2.1}$$

$f \in \mathcal{L}_K(f \in \mathcal{L}_{K_o})$. Then $\mathcal{K}_o$ is dense in $\mathcal{K}$.

In fact, assume Lemma 2.2 holds. Denote $\mathcal{C}(E) = X$ the Banach space of all E-valued continuous functions $u(\cdot)$ defined in $[t_o, t_1]$

(norm $|u(\cdot)|_X = \sup_{t_0 \le t \le t_1} |u(t)|$). Let $\epsilon > 0$, U_ϵ the $L(E,E)$-valued function defined in the square $t_0 \le s, t \le t_1$ as being equal to $U(t,s)$ in the triangle $t_0 \le s, t \le t_1$, null in the triangle $t_0 \le t \le s - \epsilon \le$ $\le t_1 - \epsilon$ and defined elsewhere such as to be continuous in the square and such that

$$\sup_{t_0 \le s, t \le t_1} |U_\epsilon(t,s)|_{L(E,E)} = C = \sup_{t_0 \le s \le t \le t_1} |U(t,s)|_{L(E,E)}. \tag{2.2}$$

It is not difficult to see that the $L(F; \mathcal{E}(E))$-valued function $N(s) =$ $= U_\epsilon(\cdot,s)B(s)$, $t_0 \le s \le t_1$ is strongly measurable and (1.5) implies that it satisfies the rest of the assumption of Lemma 2.3. Consequently Lemma 2.3 tells us that the set of elements of $\mathcal{E}(E)$ of the form

$$\int_{t_0}^{t_1} U_\epsilon(t,s)B(s)f(s)ds, \tag{2.3}$$

with $f \in \mathcal{L}_{K_0}$ is dense (in the $\mathcal{E}(E)$-topology) in the set of elements of the form (2.3) with $f \in \mathcal{L}_K$. This would yield Theorem 2.2 if we had U instead of U_ϵ in (2.3); note, however, that

$$\left|\int_{t_0}^{t} U(t,s)B(s)f(s)ds - \int_{t_0}^{t_1} U_\epsilon(t,s)B(s)f(s)ds\right|_E \le CC_1 \int_t^{\min(t+\epsilon,t_1)} \eta(s)ds$$

C_1 an upper bound for $\{|u|;\ u \in K\}$, $\eta(\cdot)$ the function in (1.5). This proves Theorem 2.2.

<u>Proof of Lemma 2.3</u>. The proof is trivial if $N(\cdot)$ is <u>uniformly</u> measurable (i.e. measurable as an $L(F;X)$-valued function). For in this case, given $\epsilon > 0$ we can find disjoint intervals whose union differs from $[t_0,t_1]$ is a set of measure $\le \epsilon$ and operators $N_k \in L(F;X)$ such that

$$|N(s) - N_k|_{L(F;X)} \le \epsilon, \ s \in I_k.$$

This makes clear that we only need to prove Lemma 2.2 for the case <u>N constant.</u> Let $f \in \mathcal{L}_K$. If $v = \int_{t_o}^{t_1} f(s)ds$, it follows from the fact that K is closed and convex that $(t_1-t_o)^{-1}v \in K$. Then it can be approximated arbitrarily well by (finite) convex combinations $\sum \lambda_k u_k$, $u_k \in K_o$. But then Nv can be approximated by elements of the form $N(t_1-t_o)u$, and $N(t_1-t_o)u = \int_{t_o}^{t_1} Nf_o(s)ds$, where $f_o(s) = u_k$ for $s \in J_k$, J_k an arbitrary family of (disjoint) subintervals of $[t_o,t_1]$, length $(J_k) = (t_1-t_o)\lambda_k$.

Observe next that if F is finite-dimensional, the concepts of strong and uniform measurability for $N(\cdot)$ coincide. We shall thus end the proof by reducing the general case to that in which $\dim F < \infty$. Let $f \in \mathcal{L}_K$. Since f is strongly measurable, we can find a g of the form

$$g(s) = \sum_{(\text{finite})} X_k(s)u_k, \tag{2.4}$$

$u_1,u_2,\ldots \in K$, $X_1X_2,\ldots$ characteristic functions of disjoint measurable sets $e_1,e_2,\ldots$ in $[t_o,t_1]$ such that $|f(s) - g(s)| \le \epsilon$ in $[t_o,t_1]$ outside a set of measure $\le \epsilon$, thus we can assume f to be of the form (2.4). Now, since each u_k can be approximated by convex combinations $\sum_{j=1}^{m(k)} \lambda_{kj}u_{kj}$, $u_{kj} \in K_o$, we can assume the values of f actually belong to the convex hull K' of the points u_{kj}, $k = 1,2,\ldots$ $1 \le j \le m(k)$. But K' is contained in the finite-dimensional subspace F' of F generated by the

u_{kj}, and those points satisfy Assumption 1 (with respect to K') thus our result for finite-dimensional F applies, completing the proof of Lemma 2.2.

2.4. REMARK. Assume F is the dual of another Banach space F_1. Thus K is compact in F with respect to the weak topology. But then, by the Krein-Milman theorem ([1], Chapter V, 8.4), K is the closed (in the weak topology) convex envelope of its set of extremal points. However, the closed convex envelope of a set is the same in the strong as in the weak topology, thus K_e, the set of extremal points of K satisfies Assumption 2.1. In some cases, K_o can be chosen to be a proper subset of K_e. The most interesting case in application is that in which K_o is substantially smaller than K; for instance, if K is a polyhedron in a finite-dimensional space F, we may take K_o to be the set of its vertices. Thus, the steering of (1.1) can be achieved up to any degree of accuracy with controls assuming only a finite number of values.

2.5. REMARK. Theorem 2.2 admits evident generalizations to infinite time intervals (t_o,∞) moving control sets $K = K(t)$, etc.

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FOOTNOTES

1. See [1] for definitions and results used here.

2. All the integrals throughout this paper are Bochner integrals; see [6], Chapter 3 for an exposition of the theory of integration of vector-valued functions.